Classical probabilities

Markov chains

Markov chains?

Andrei A. Markov
Russian mathematician 1856-1922

Markov chains as pedagogic, analytical and prognostic tool

1. Helps us understand probabilities and their additions and co-variations

2. Helps us analyse data in a new and interesting way

3. Might not stand up on its own, but provides a good complement to traditional statistical post-processing

A transition table for rain fall at Stockholm-Arlanda airport (100 typical cases)

Following 6 hours

First 6 hours

	rain	dry	
rain	**17**	**11**	28 cases
dry	**11**	**61**	72 cases
	28 cases	72 cases	

62% (17/28) probability of continued rain

38% probability of rain to dry

15% (11/72) probability of dry to rain

85% (61/72) probability of staying dry

We note that the climatological rain probability is **28%**

Transition probabilities

62% probability of continued rain

38% probability of rain to dry

Next 6 hours

Last 6 hours

$$
\begin{array}{c|cc}
 & \mathbf{R} & \otimes \\
\hline
\mathbf{R} & .62 & .38 \\
\otimes & .15 & .85
\end{array}
$$

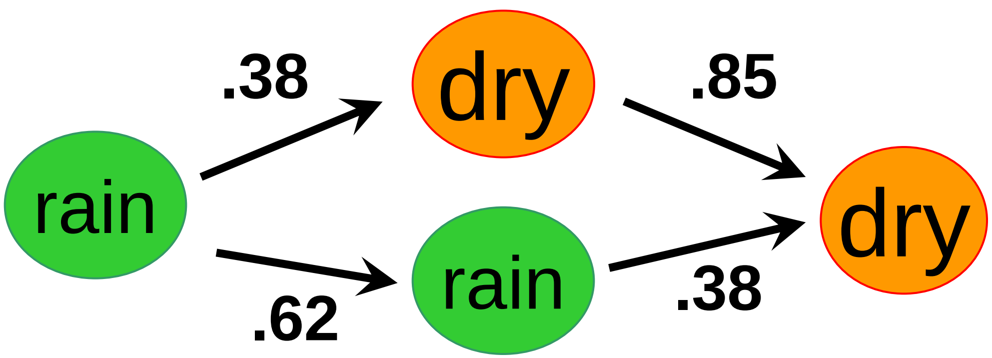
15% (11/72) probability of dry to rain

85% (61/72) probability of staying dry

$$\mathbf{R} = \text{rain} \qquad \otimes = \text{dry}$$

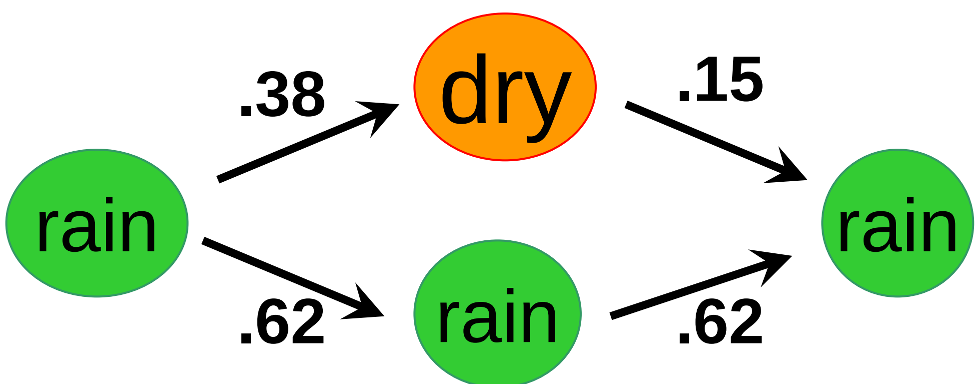
$$\begin{array}{c} \\ \mathbf{R} \\ \otimes \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{R} & \otimes \end{array} \\ \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \end{array}$$

What can we say about the 12 h transition? From 00-06 to 12-18?



$$0.38\cdot0.85+0.62\cdot0.38=0.56$$

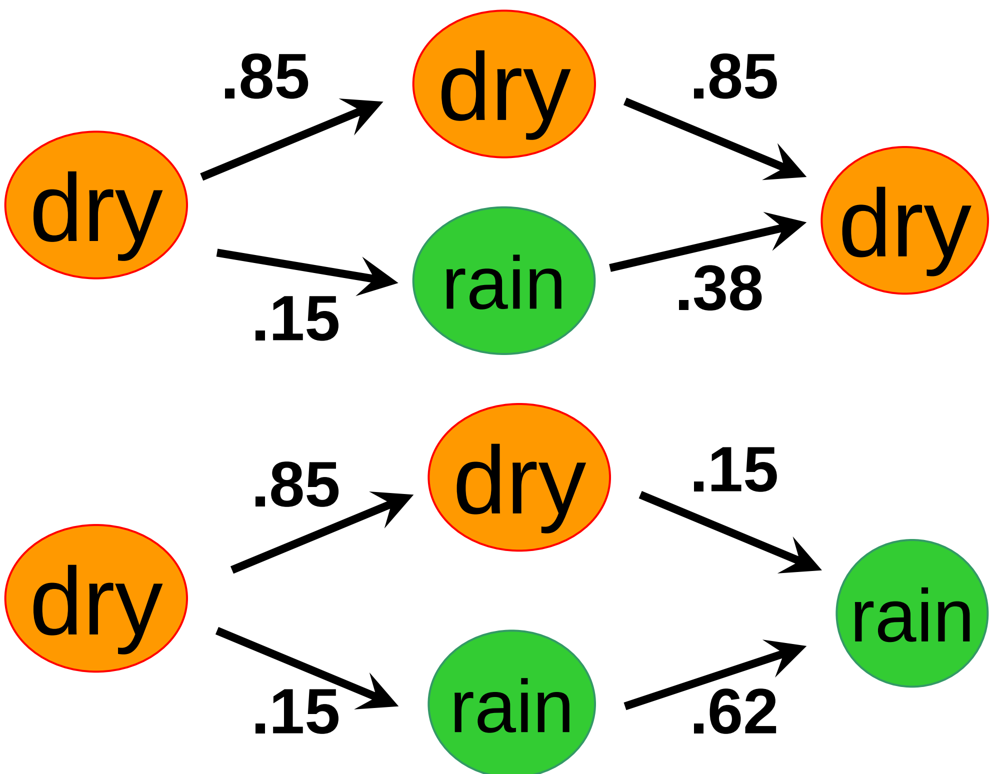
$$0.38\cdot0.15+0.62\cdot0.62=0.44$$



$$\begin{array}{c} \mathbf{R} \\ \otimes \end{array} \begin{pmatrix} .44 & .56 \\ . & \end{pmatrix}$$

$$\begin{array}{c} \\ \mathbf{R} \\ \otimes \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{R} & \otimes \end{array} \\ \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \end{array}$$

What can we say about the 12 h transition? From 00-06 to 12-18?



$$0.85 \cdot 0.85 + 0.62 \cdot 0.38 = 0.78$$

$$0.85 \cdot 0.15 + 0.62 \cdot 0.62 = 0.22$$

$$\begin{array}{c} \\ \mathbf{R} \\ \otimes \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{R} & \otimes \end{array} \\ \begin{pmatrix} .44 & .56 \\ .22 & .78 \end{pmatrix} \end{array}$$

The matrice can also be regarded as an **algebraic transition matrix** $\mathbf{R} = \text{rain} \quad \otimes = \text{dry}$

Next 6 hours

Last 6 hours

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{R} & \otimes \end{array} \\ \begin{array}{c} \mathbf{R} \\ \otimes \end{array} & \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \end{array}$$

Then the full force of **3000 years of algebra** can be applied, in particular matrix algebra

Some matrice algebra

A brief repetition of matrix algebra...

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ - \end{pmatrix} = \begin{pmatrix} 17 \\ - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

A brief repetition of matrix algebra...

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2\cdot 5 & - \end{pmatrix} = \begin{pmatrix} 13 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} - & 1\cdot 4 + 2\cdot 6 \end{pmatrix} = \begin{pmatrix} 13 & 16 \end{pmatrix}$$

A brief repetition of matrix algebra...

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2\cdot 5 & - \\ - & - \end{pmatrix} = \begin{pmatrix} 13 & - \\ - & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} - & 1\cdot 4 + 2\cdot 6 \\ - & - \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ - & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} - & - \\ - & 3\cdot 4 + 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ 29 & 36 \end{pmatrix}$$

Matrix multiplication yields a forecast

"Forecast" 12-18 h later

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}^2 = \begin{pmatrix} .44 & .56 \\ .22 & .78 \end{pmatrix}$$

Observed 12-18 h later

Verification Last 6 hours

$$\begin{pmatrix} .47 & .53 \\ .21 & .79 \end{pmatrix}$$

The matrix multiplication continues...

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}^4 = \begin{pmatrix} .33 & .67 \\ .26 & .74 \end{pmatrix}$$

Next 18-24 hours

verification → Last 6 hours

$$\begin{pmatrix} .39 & .61 \\ .24 & .76 \end{pmatrix}$$

After repeated multiplications the values converge towards the ”climate”

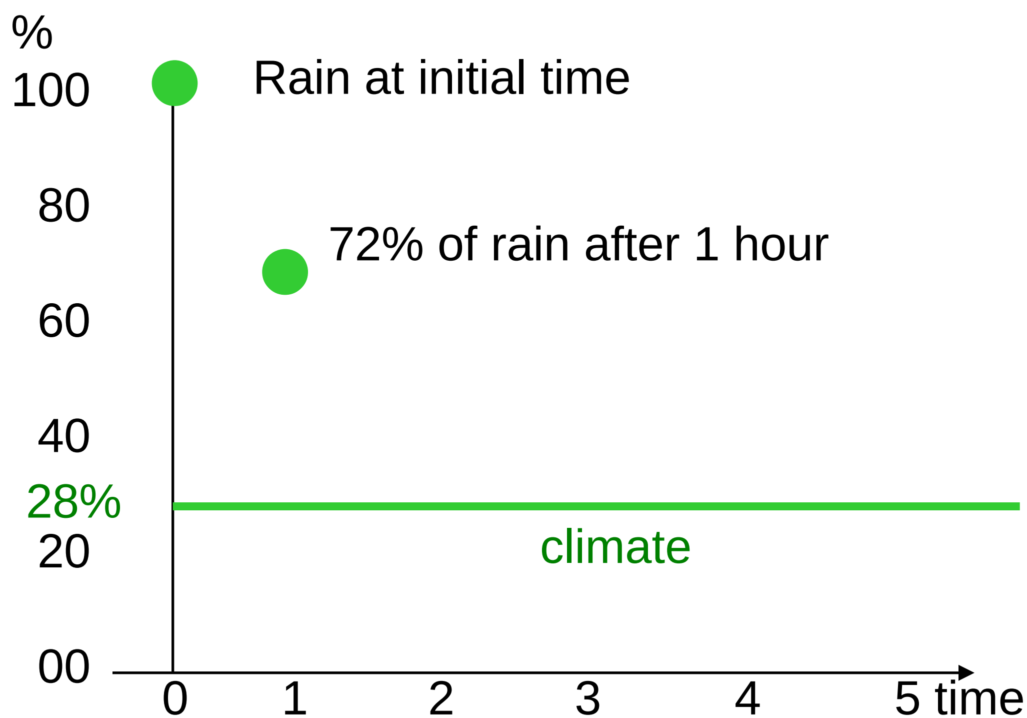
$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}^8 = \begin{pmatrix} .72 & .28 \\ .72 & .28 \end{pmatrix}$$

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}^\infty = \begin{pmatrix} .72 & .28 \\ .72 & .28 \end{pmatrix}$$

Matrix = climate matrix + residual

Persistence Variability

$$\begin{pmatrix} .85 & .15 \\ .38 & .62 \end{pmatrix}^N = \begin{pmatrix} .72 & .28 \\ .72 & .28 \end{pmatrix} + 0.47^N \begin{pmatrix} .28 & -.28 \\ -.72 & .72 \end{pmatrix}$$

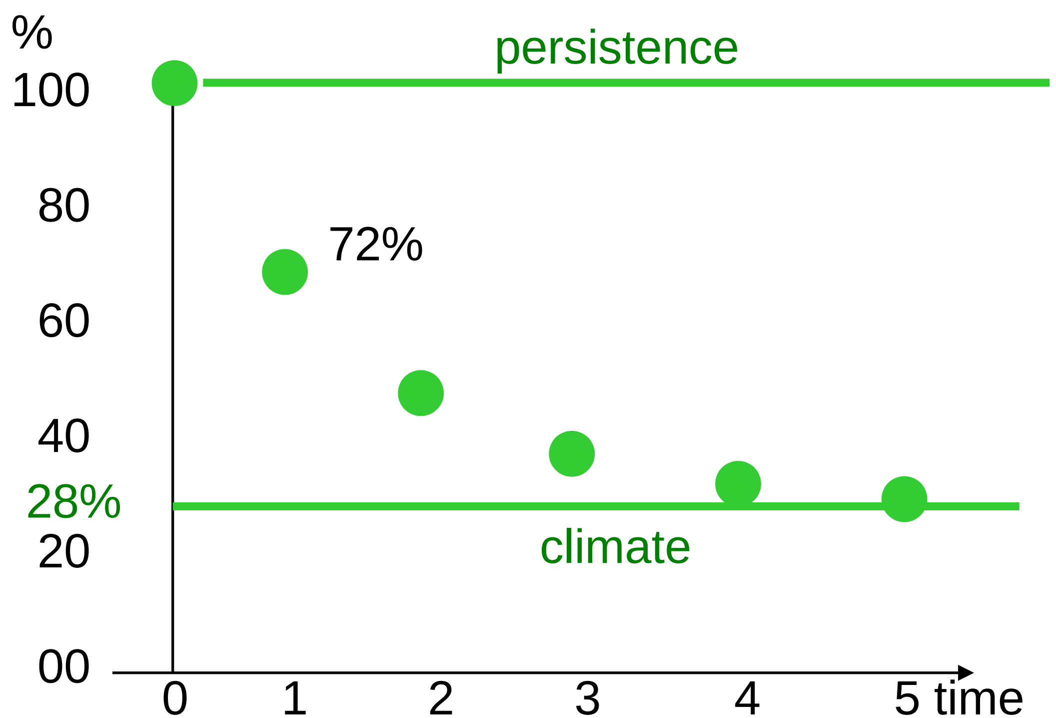


Matrix = climate matrix + residual

$$\begin{pmatrix} .85 & .15 \\ .38 & .62 \end{pmatrix}^N = \underbrace{\begin{pmatrix} .72 & .28 \\ .72 & .28 \end{pmatrix}}_{\text{Persistence}} + 0.47^N \underbrace{\begin{pmatrix} .28 & -.28 \\ -.72 & .72 \end{pmatrix}}_{\text{Variability}}$$

$$\begin{pmatrix} .85 & .15 \\ .38 & .62 \end{pmatrix}^\infty = \begin{pmatrix} .72 & .28 \\ .72 & .28 \end{pmatrix}$$

Approach to clime
for increasing time



Similarities with forecast models:

Obs/analysis → Model → Forecast

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}.62 & .38\\ .15 & .85\end{pmatrix} = \begin{pmatrix}.62 & .38\end{pmatrix}$$

Rain

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}.62 & .38\\ .15 & .85\end{pmatrix} = \begin{pmatrix}.15 & .85\end{pmatrix}$$

Dry

$$\begin{pmatrix}.70 & .30\end{pmatrix}\begin{pmatrix}.62 & .38\\ .15 & .85\end{pmatrix} = \begin{pmatrix}.48 & .52\end{pmatrix}$$

70% rain 30% dry

…more examples:

Obs/analysis → Model → Forecast

$$\begin{pmatrix} .70 & .30 \end{pmatrix} \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \quad \begin{pmatrix} .48 & .52 \end{pmatrix}$$

$$\begin{pmatrix} .90 & .10 \end{pmatrix} \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \quad \begin{pmatrix} .58 & .42 \end{pmatrix}$$

$$\begin{pmatrix} .30 & .70 \end{pmatrix} \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \quad \begin{pmatrix} .29 & .71 \end{pmatrix}$$

..which is almost identical with the input vector

Left eigenvector and -value

$$\begin{pmatrix} .28 & .72 \end{pmatrix} \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} =$$

$$= \begin{pmatrix} .28 & .72 \end{pmatrix} = 1 \cdot \begin{pmatrix} .28 & .72 \end{pmatrix}$$

$$\begin{pmatrix} -.28 & .28 \end{pmatrix} \begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} =$$

eigenvalues

$$= \begin{pmatrix} -.13 & .13 \end{pmatrix} = 0.47 \begin{pmatrix} -.28 & .28 \end{pmatrix}$$

Right eigenvector and -value

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvalues

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix} \begin{pmatrix} -2.56 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.21 \\ 0.47 \end{pmatrix} = 0.47 \begin{pmatrix} -2.56 \\ 1 \end{pmatrix}$$

The initial transition matrix can be decomposed into a weighted sum of two new matrices

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}^n = \begin{pmatrix} .28 & .72 \\ .28 & .72 \end{pmatrix} + \underbrace{0.47}_{\text{Eigen value}}{}^n \cdot \begin{pmatrix} .72 & -.72 \\ -.28 & .28 \end{pmatrix}$$

Persistence probabilities

Climate probabilities

Time folding factor

Variability

Meteorological interpretation

$$\begin{pmatrix} .62 & .38 \\ .15 & .85 \end{pmatrix}$$

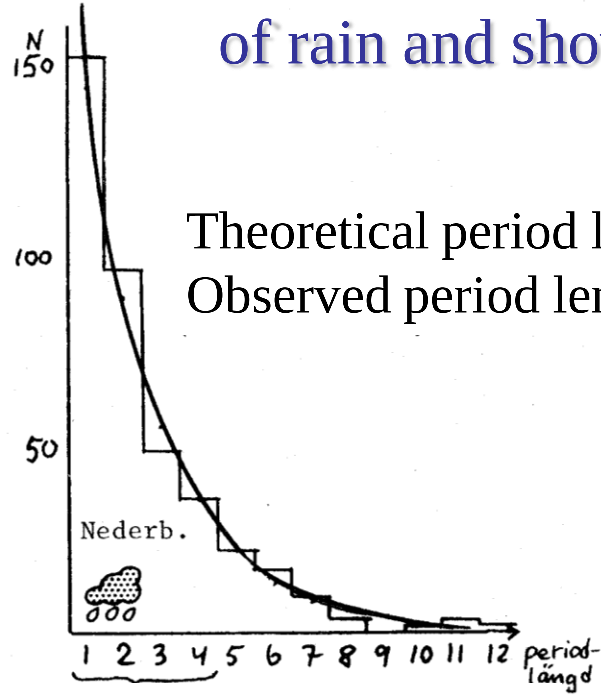
Mean period length =

$(1 - 0.62)^{-1} \cdot 6$ hours = 15.8 hours

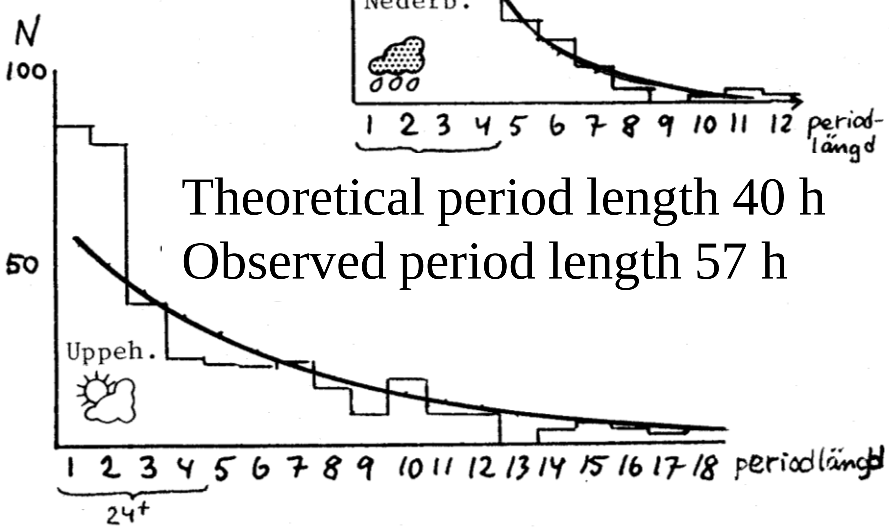
Generally (τ) with p_{ii} as the persistence probability of the class:

$$\tau = \frac{\Delta t}{(1 - p_{ii})}$$

Calculated and observed duration of rain and showers

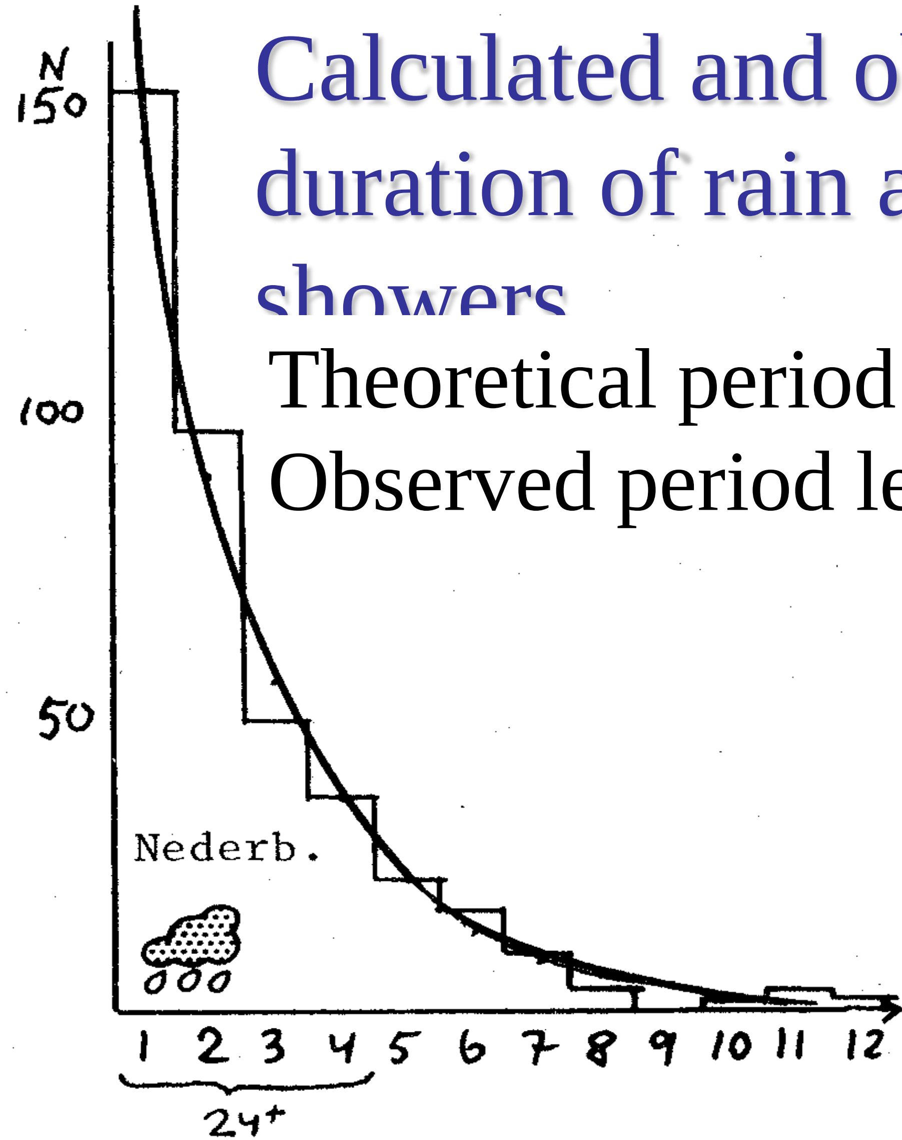


Theoretical period length 15.8 h
Observed period length 15.9 h



Theoretical period length 40 h
Observed period length 57 h

Calculated and observed duration of rain and showers



Number of 6 h periods

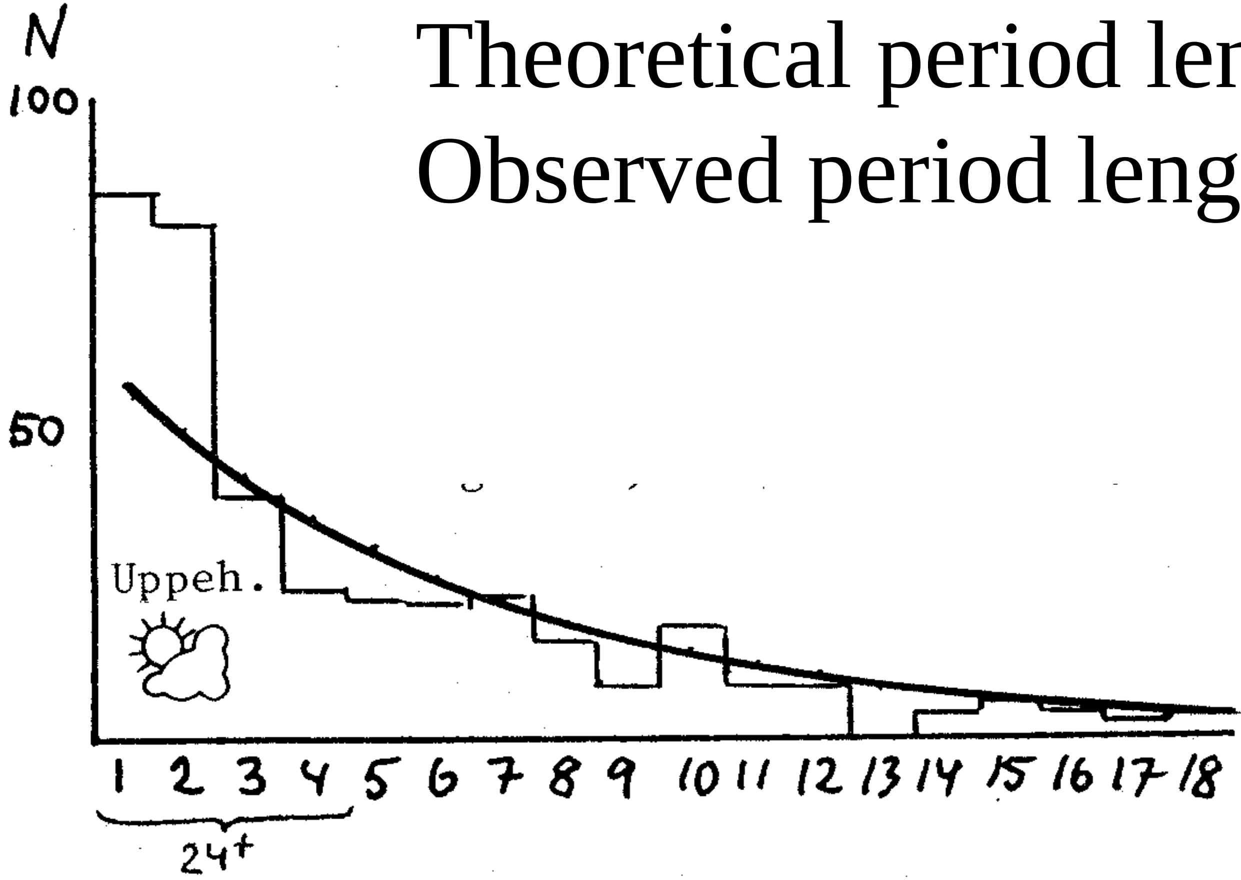
Theoretical period length 15.8 h

Observed period length 15.9 h

Calculated and observed duration of dry weather

Theoretical period length 40 h

Observed period length 57 h



Number of 6 h periods

Three state transitions

∇ = *showers*

$$\begin{array}{c} \\ \otimes \\ \bullet \\ \nabla \end{array}\begin{array}{c} \begin{array}{ccc} \otimes & \bullet & \nabla \end{array} \\ \begin{pmatrix} .85 & .07 & .08 \\ .28 & .59 & .13 \\ .50 & .11 & .39 \end{pmatrix} \end{array} = \begin{pmatrix} .72 & .15 & .13 \\ .72 & .15 & .13 \\ .72 & .15 & .13 \end{pmatrix}$$

$$+0.53^N \begin{pmatrix} .21 & -.17 & -.04 \\ -.96 & .79 & .17 \\ -.03 & .02 & .01 \end{pmatrix}$$

$$+0.30^N \begin{pmatrix} .07 & -.02 & -.09 \\ .24 & .06 & -.30 \\ -.69 & .02 & .87 \end{pmatrix}$$

Rain persistence (.79)

Shower persistence (.87)

6 hour transition matrix for Arlanda airport weather

		O	Θ	⊕	•	∇
0 – 2 / 8	O	.61	.28	.06	.02	.03
3 – 5 / 8	Θ	.24	.40	.20	.07	.09
6 – 8 / 8	⊕	.05	.23	.47	.13	.13
	•	.03	.12	.13	.59	.13
	∇	.15	.19	.16	.11	.39

Climate matrix

	O	Θ	⊕	●	∇
O	.25	.27	.20	.15	.13
Θ	.25	.27	.20	.15	.13
⊕	.25	.27	.20	.15	.13
●	.25	.27	.20	.15	.13
∇	.25	.27	.27	.15	.13

$$\begin{array}{ccccc} \bigcirc & \Theta & \oplus & \bullet & \nabla \end{array}$$

$$0.61^N \begin{pmatrix} \mathbf{.44} & .16 & -.09 & -.37 & -.14 \\ .09 & .03 & -.02 & -.08 & -.03 \\ -.23 & -.08 & .05 & .19 & .07 \\ -.55 & -.20 & .12 & \mathbf{.46} & .17 \\ -.06 & -.02 & .01 & .05 & .02 \end{pmatrix}$$

Persistence for clear sky and rain

$$\begin{array}{ccccc} \bigcirc & \Theta & \oplus & \bullet & \nabla \end{array}$$

$$0.41^N \begin{pmatrix} .17 & -.06 & -.25 & .24 & -.09 \\ -.07 & .03 & .11 & -.10 & .04 \\ -.27 & .10 & \mathbf{.40} & -.14 & .15 \\ .26 & -.10 & -.39 & \mathbf{.37} & -.14 \\ -.07 & .03 & .10 & -.10 & .04 \end{pmatrix}$$

Persistence of overcast and rain

$$\begin{array}{ccccc} \bigcirc & \Theta & \oplus & \bullet & \nabla \end{array}$$

$$0.29^N \begin{pmatrix} -.01 & -.03 & -.04 & -.01 & .09 \\ .02 & .04 & .05 & .01 & -.12 \\ .06 & .12 & .14 & .04 & -.36 \\ .02 & .05 & .06 & .02 & -.15 \\ -.13 & -.27 & -.32 & -.10 & \mathbf{.82} \end{pmatrix}$$

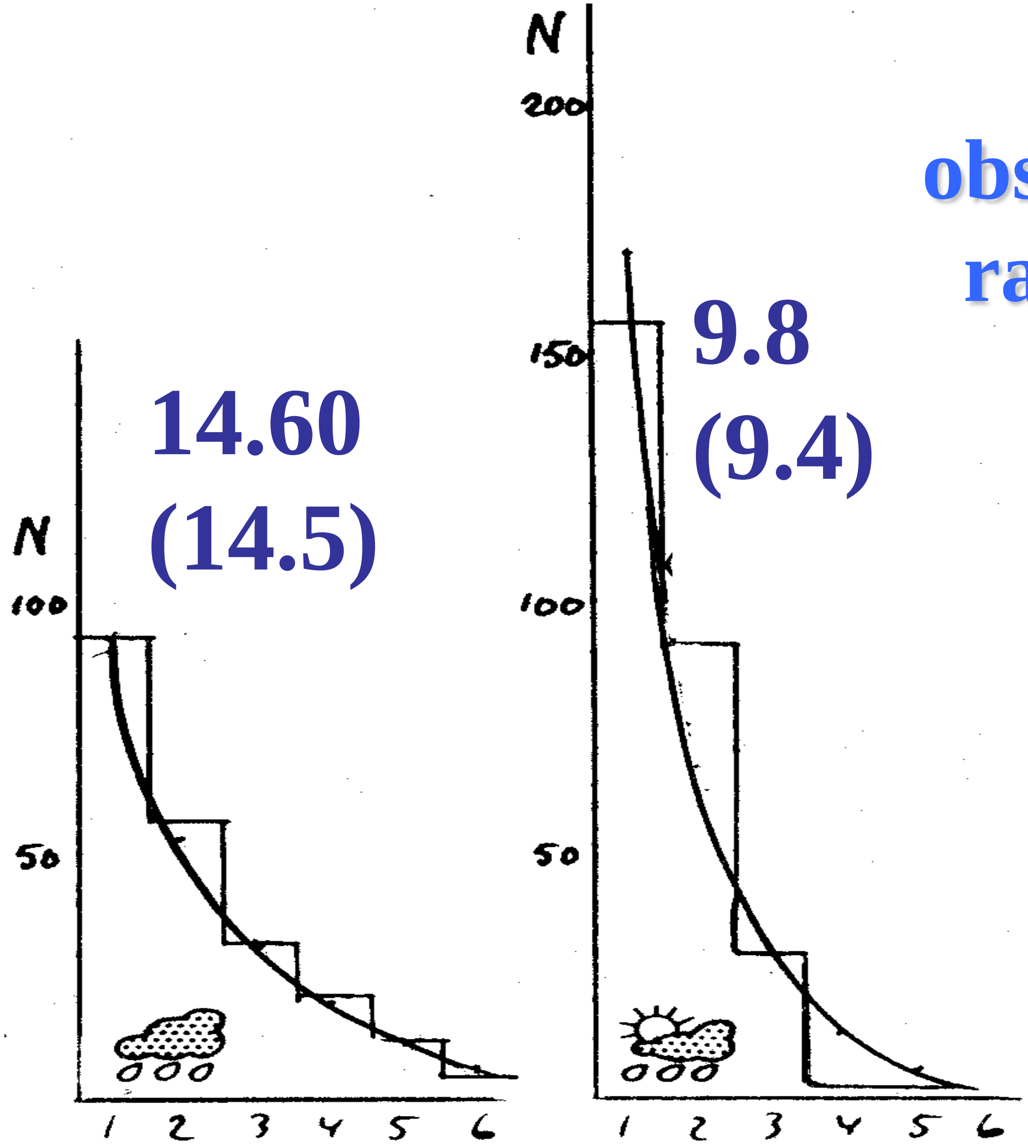
Persistence of showers

$$\begin{array}{ccccc} \bigcirc & \Theta & \oplus & \bullet & \nabla \end{array}$$

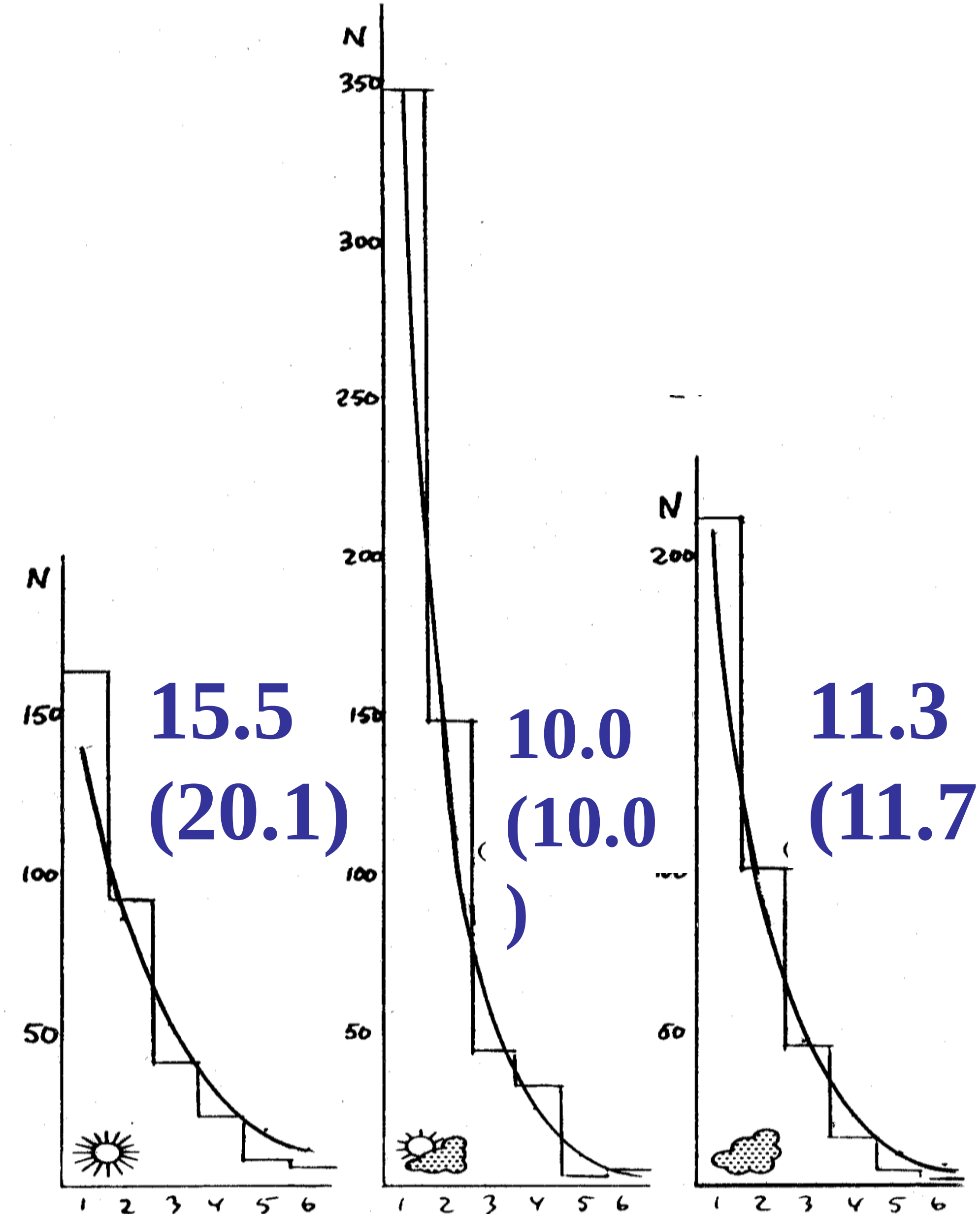
$$0.15^N \begin{pmatrix} .16 & -.33 & .18 & .01 & .01 \\ -.29 & \mathbf{.63} & -.34 & .02 & .04 \\ .19 & -.40 & .21 & -.01 & .01 \\ .01 & -.02 & .01 & .00 & .00 \\ .00 & -.01 & .00 & .00 & .00 \end{pmatrix}$$

Persistence of variable sky

Calculated and observed duration rain and showers



Calculated and observed duration of clear, variable and overcast sky



15.5 (20.1)

10.0 (10.0)

11.3 (11.7)

END