Proving that the Earth rotates by measuring the deflection of objects dropped in a deep mine

The French-German mathematical contest between Pierre Simon de Laplace and Friedrich Gauß 1803

by Anders Persson
FRMetS (Fellow of the British Royal Meteorological Society)
University of Uppsala (Sweden)

1. From Aristotle to Galileo

From the 17th century well into the 19th century the deflection of falling objects was a hotly debated subject, first as a way to prove or disprove the Copernican theory, later about the details of the deflection. This paper from 1803 *"Mémoire sur le mouvement d'un corps qui tombe d'une grande hauteur"* by Laplace is a scientific milestone in this debate:

- It was written to predict the likely deflection of objects dropped in a mine shaft for a campaign to prove the rotation of the Earth;
- It was made in an unofficial competition with the renowned German mathematician Friedrich Gauß who had got the same assignment ;
- Both arrived at the correct result and were thereby the first scientists to derive and physically interpret what was later to become known as the "Coriolis Effect".

The experiments of deflection of falling bodies was an answer to a problem already posed by Aristotle and repeated by anti-Copernicans: if the Earth was spinning around its axis an object dropped from a tower would be "left behind", i.e. deflected to the ***west***[1]. In Galileo's (alleged) experiment at the tower of Pisa the objects landed at the base with no obvious deflection. But this was, according to Galileo and other supporters of the Copernican model, not an indication that the Earth did not move since the objects took part in the Earth's rotation. As everybody could see: an object dropped from a mast on a sailing ship in full motion would also land at the foot of the mast.

1. Earth spins around its axis from west to east.

2. The naïve model

The debate could have ended here if the anti-Copernicans had based all their opposition on a possible westerly deflection. What the scientists among them actually claimed was that objects would move *differently* depending on if the Earth was moving or not. And here they could score several points. Galileo had himself admitted in his famous *Dialogo sopra i due sistemi del mondo* in 1632 that since the top of the tower was further away from the Earth's centre, the rotational velocity at the top would be slightly larger than at the surface. An object falling from the top would therefore overtake the tower and land slightly ahead, to the ***east*** of it.

If we put this reasoning into mathematics, we will find that an object dropped in northern Italy (at 43° N) from a tower of 100 m height will take 4.515 seconds to reach the ground. Since the rotational velocity at the top is 5.3 mm/sec faster than at the surface, the object will, when it reaches the surface, have "moved ahead" by 24 mm. (fig.1).

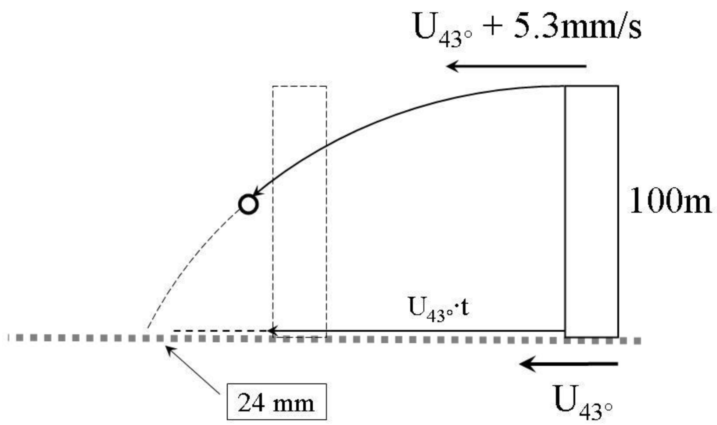

***Fig. 1:* A simplistic model of the deflection of vertically falling objects.** *Because the top of a tall building due to the earth's rotation moves faster than the bottom of the building, an object dropped from the top will, with its slightly higher horizontal velocity, reach the surface ahead of an object moving with the surface velocity.*

Such small deviations were at Galileo's time difficult to confirm by measurements. Even worse, the deflection according to this simplistic model is not quite correct and yields results which are 50% too large. This is because we have ignored the curvature of the earth. During the 4.5 seconds the object falls, the tower and its environment move 1½ kilometre to the east. Not a long distance, but the curvature of the Earth's surface, although very small, cannot be neglected. It is to their scientific credit that both the Copernicans and the anti-

Copernicans realised this and the debate concentrated on the kind of trajectory the falling object would follow.

3. THE ITALIAN DEBATE

To simplify the investigation they chose to focus on objects dropped from a tower at the equator. To allow the dropped objects to be free to continue their fall also after they had reached a distance to the earth centre equal to its radius (=surface) they designed a quite ingenious mental picture of the Earth at the equator halved in two hemispheres.

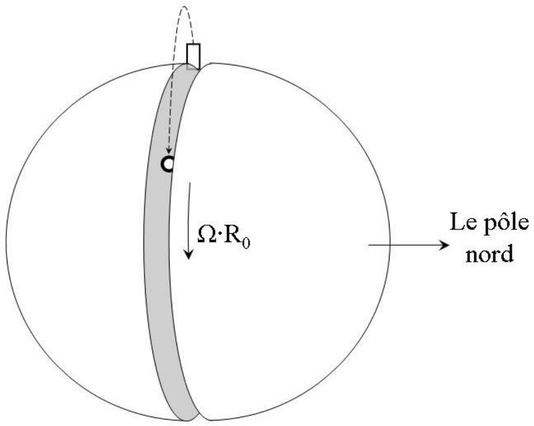

***Figure 2:* In order to allow the falling bodies to continue their journey the Italian scientists imagined the Earth as two separate hemispheres.**

With the underlying ground no longer flat but spherically curved the trajectory could no longer be a parabola (or part of a parabola). So what was it? In his *Dialogue* 1632 Galileo suggested a semicircle with a diameter equal to the radius of the Earth. This was a pure speculation on his side, had no basis in his mechanical theories and was also against the generally accepted view that the object would follow a spiral trajectory, most likely an Archimedean spiral trajectory towards the Earth's centre. But five years later he changed his mind and agreed in *Dialogue* 1638 with the prevailing view. Both sides were in full agreement that the object would come to rest at the centre of the Earth.

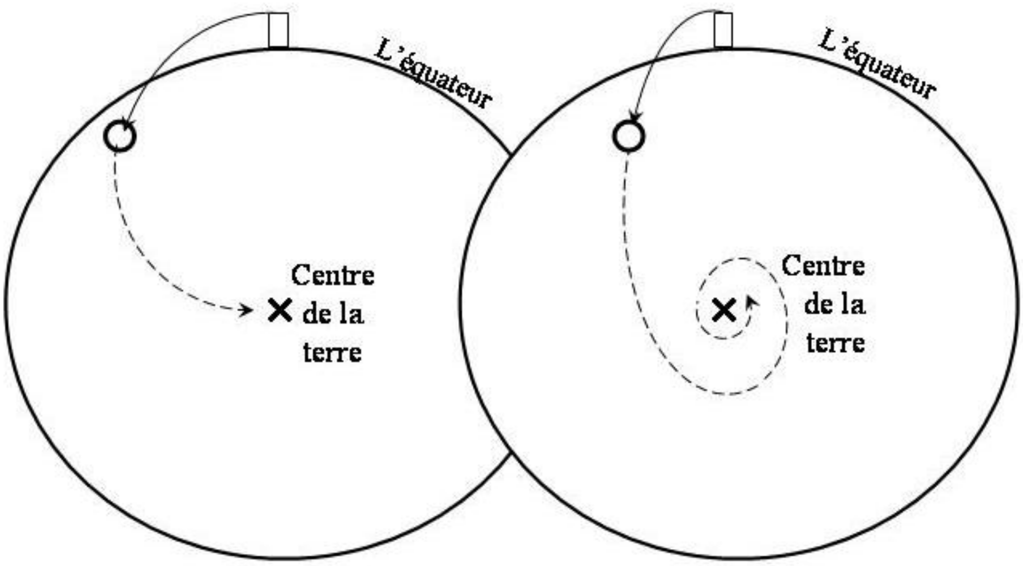

***Figure 3:* It was taken as an axiom by all scientists in the early 17th century that an object, able to penetrate the earth without any loss of speed, would end up at the centre of the Earth**. *The only disagreement was the type of trajectory that would carry it to the centre. Galileo initially suggested a semi-circle (left) against the prevailing opinion which favoured a spiral, most likely an Archimedean spiral (right).*

The debate that followed has been taken by modern historians of science as a textbook example of how the tremendous power of "mental or intellectual inertia" and of the very slow and gradual way in which even "superior minds" succeed in liberating themselves from the traditional and habitual, what the English 17th century philosopher Francis Bacon called "*idola tribus*". Because what everybody agreed on, that the falling body would end up in the centre of the earth, *was fundamentally wrong*! It was fundamentally wrong because it effectively prevented the scientists to even reach approximate or qualitatively correct solutions!

The debate therefore became very confused. So for example in 1667 Giovanni Alfonso Borelli (1608-1679) put forward a hypothesis that curved motion was composed of one rectilinear tangential motion and one accelerated towards the centre. We now know this is perfectly true, but his idea was rejected, also by himself and his followers, when it was found that the falling object would then not reach the centre of the Earth!

4. NEWTON'S "FANCY"

The Italian debate quickly "spilled over" to England. David Gregory (1659-1708), one of the members of the Royal Society, reported this Italian debate in 1668. It fuelled the interest in the problem also in England, and in 1674 Robert Hooke published a book with the title *"An Attempt to Prove the Motion of the Earth"*. One of the methods he suggested was to observe the deflection of objects dropped from high buildings. He had also qualitatively foreseen Newton's laws in *Principia*:

- All celestial bodies have an attraction or gravitational power towards their own centres;
- A body put in motion in a straight line will continue to move until it by some power is deflected into a curved motion;
- The nearer the object the stronger this attractive power.

Hooke had thereby conceived gravity as an attractive force drawing a body downward, rather than being an Aristotelian "tendency to fall" within the body itself. Hooke's profound physical intuition was acquired through numerous experiments, perhaps several hundreds. His remarkable physical intuition and understanding was based on mechanical analogues rather than mathematical reasoning and would have a crucial importance to the development of Newton's thinking.

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In November 1679 Robert Hooke, in his capacity as newly elected Secretary of the Royal Society, wrote a letter to Isaac Newton. His intention was to draw Newton into a discussion on planetary motion, in particular the reason for the elliptical orbits of the planets. But Newton had something else on his mind, what he called "a fancy of my own": *the deflection of objects dropped from a high altitude* as proof of the Earth's rotation.

Much later in his life Isaac Newton would tell his friends that it was in 1666, while watching apples fall from the tree in his family garden, that made him speculate about earthly bodies and the moon being attracted by the same gravitational forces. What has made scholars skeptical is that it was in 1726, sixty years after the alleged event, and at a time when Newton was engaged in priority argument with other scientists. In 1666 he was developing ideas in mathematics and optics, and there is no documentary evidence about deeper thoughts on the nature of gravitation. We also know from the history of science that such profound theoretical concepts rarely appear "out of the blue".

However, if we place the "falling apple event" thirteen years later, in 1679, it gains much more credibility. Newton had spent most of that summer and autumn at his family home in Lincolnshire. His mother had just died and he had to attend to family matters. There had been a lot of opportunities to see apples fall in the family garden.

5. THE ELLIPTIC PATH

The exchange of letters that followed with Hooke during the winter 1679-80 shows that Newton had not yet achieved a deeper understanding of celestial mechanics. His first idea was to suggest, like the general scientific opinion of the time, that a falling object would, in principle, approach the centre of the earth in a spiral. Thanks to Hooke he came to realize, that it instead would rather follow an elliptic path (figure 4).

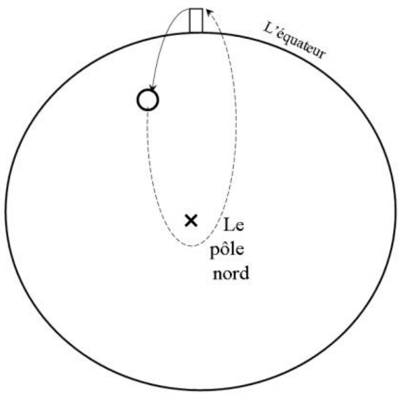

***Figure 4:* Hooke's and Newton's agreed opinion about the elliptical trajectory of a falling body within the earth's gravitational attraction.**

From this insight, that a falling object follows the same type of orbit as any of the planets around the Sun, it is not far-fetched to infer that the motions of these different bodies might be controlled by the same mechanism – universal gravitation. Still, even for a genius like Newton, it took a few more years to lead into conclusions in "*Principia*".

6. Gauß and Laplace

More than a century later there was a renewed interest in the problem of the deflection of falling objects. The explanation of the flattening of the poles proved that the earth rotated and settled the dispute among scientists. But it was in some way an indirect proof.

In 1803 an experiment, dropping iron pebbles in a 90 metre deep mineshaft, was conducted in Germany. The event attracted the interest of the scientific community and the 24-year German mathematician Carl Friedrich Gauß and the 53-year French mathematician Pierre Simon de Laplace volunteered to calculate the theoretically expected deflection.

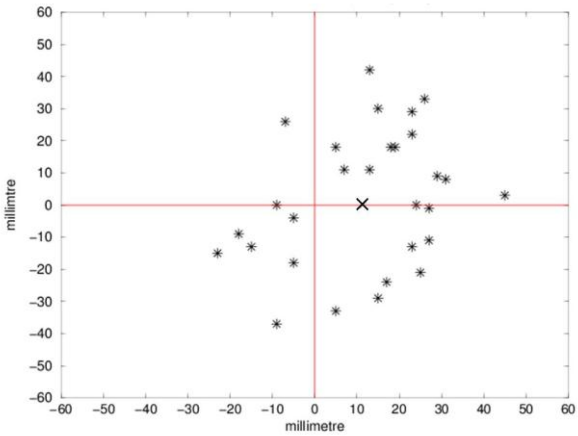

***Fig.5:* The hit-picture from the Schlebusch experiment 1803.** *A cross marks the theoretically derived deflection.*

There was an element of competition since Gauß had the year before managed to calculate the orbit of the newly discovered asteroid Ceres, something Laplace had deemed impossible. Both came up with the right answer by deriving the full three-dimensional equation for motions on a rotating earth.

They specifically pointed out that the Coriolis-terms (as we call them)[2] were responsible for the deflection. Gauß and Laplace were thereby the first scientists to contribute to the proof of the rotation of the earth some 50 years before Foucault's famous pendulum experiment and to analyse correctly the relative motion in connection with rotation, 30 years before Coriolis' mathematical paper (1835).

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2. About Coriolis's 1835 text, see *BibNum* analysis by A. Moatti, October 2011 (online).

7. LAPLACE'S DERIVATION – AND THE MODERN ONE

What makes Laplace's derivation difficult to follow for modern readers are

- Lack of graphical illustrations. In figure 6, I have made an attempt to show visually what Laplace might have had in mind.
- The mathematical notations of the time. Laplace for example defines the latitude as the angle from the earth's rotational axis (co-latitude) instead as from the equatorial plane as we now do.
- The use of Cartesian component forms. During the 19th century British, German and American physicists (not mathematicians) developed for practical reasons, to facilitate intuitive interpretations, the modern vectorial system.
- In order to acquire as exact an estimation as possible, Laplace also wanted to take the possible effect of the air resistance into account.
- Laplace wanted also, with as detailed calculations as possible, try to find if there was a possible minor southerly deflection.

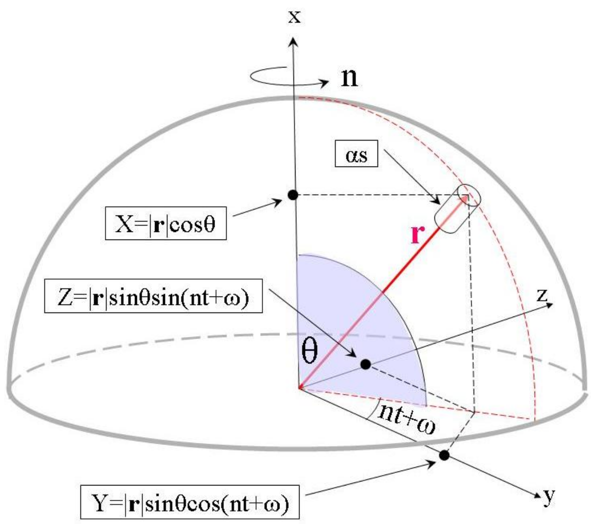

***Figure 6:* Laplace's computational model of the Earth with a tower at co-latitude θ (corresponding to latitude 90-θ).**

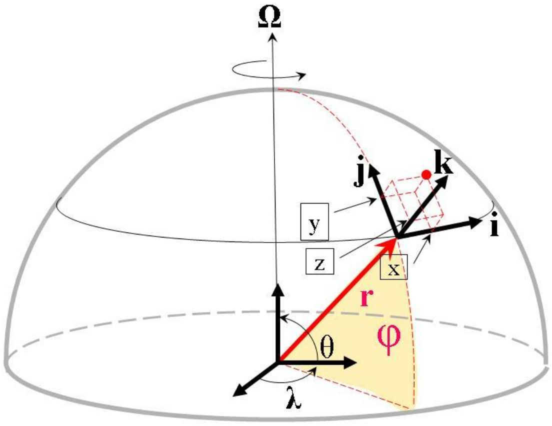

***Figure 7*: The modern spherical coordinate system with θ, λ and r as axes.** *Locally a Cartesian* x, y *and* z *coordinate system can be defined.*

Laplace derivation does essentially the same as today's derivations, a coordinate transformation from an absolute frame of reference to a relative, rotating one. The rate of change of vector **A** as observed in the absolute frame of reference (what Laplace marked x, y and z) and in the relative frame of reference (by Laplace noted as reference frame X,Y and Z) rotating with an angular velocity **Ω** is described by the very powerful relation :

$$\left(\frac{\mathbf{dA}}{\mathbf{dt}}\right)_{abs} = \left(\frac{\mathbf{dA}}{\mathbf{dt}}\right)_{rel} + \mathbf{\Omega}\times\mathbf{A} \qquad (1)$$

A can be any vector and we choose it to be the position vector **r**

$$\left(\frac{\mathbf{dr}}{\mathbf{dt}}\right)_{abs} = \left(\frac{\mathbf{dr}}{\mathbf{dt}}\right)_{rel} + \mathbf{\Omega}\times\mathbf{r} \qquad (2a)$$

or with **v** denoting the velocity

$$\mathbf{v}_{abs} = \mathbf{v}_{rel} + \mathbf{\Omega}\times\mathbf{r} \qquad (2b)$$

We then apply *(1)* on the absolute velocity which yields

$$\left(\frac{\mathbf{dv}_{abs}}{\mathbf{dt}}\right)_{abs} = \left(\frac{\mathbf{dv}_{abs}}{\mathbf{dt}}\right)_{rel} + \mathbf{\Omega}\times\mathbf{v}_{abs} \qquad (3)$$

Substituting *(2b)* into the right hand side of *(3)* gives

$$\left(\frac{\mathbf{dv}_{abs}}{\mathbf{dt}}\right)_{abs} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{v}_{rel} + \mathbf{\Omega}\times\mathbf{r}\right)_{rel} + \mathbf{\Omega}\times\left(\mathbf{v}_{rel} + \mathbf{\Omega}\times\mathbf{r}\right) \qquad (4)$$

which yields

$$\left(\frac{\mathbf{dv}_{abs}}{\mathbf{dt}}\right)_{abs} = \left(\frac{\mathbf{dv}_{rel}}{\mathbf{dt}}\right)_{rel} + 2\mathbf{\Omega}\times\mathbf{v}_{rel} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \qquad (5)$$

As seen in *an absolute frame of reference:*

$$\mathbf{acc}_{abs} = \mathbf{acc}_{rel} + 2\mathbf{\Omega}\times\mathbf{v}_{rel} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \qquad (6a)$$

We are, however, interested in the accelerations seen from the *relative frame of reference:*

$$\mathbf{acc}_{rel} = \mathbf{acc}_{abs} - 2\mathbf{\Omega}\times\mathbf{v}_{rel} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \qquad (6b)$$

where $-2\,\mathbf{\Omega}\times\mathbf{v}_{rel}$ is the *Coriolis force (per unit mass)* pointing to the right of the motion and $-\,\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$ the *centrifugal acceleration* pointing outward, from the axis of rotation.

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Algebraically the three-dimensional components can be found by applying standard mathematics for cross products vectors

$$-2\mathbf{\Omega}\times\mathbf{v}_{rel} = -2\Omega\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \cos\varphi & \sin\varphi\\ u_{rel} & v_{rel} & w_{rel}\end{vmatrix} = \qquad (7)$$

$$-(2\Omega w_{rel}\cos\varphi - 2\Omega v_{rel}\sin\varphi)\,\mathbf{i} - 2\Omega u_{rel}\sin\varphi\,\mathbf{j} + 2\Omega u_{rel}\cos\varphi\,\mathbf{k}$$

which yields the deflection of east-west or latitudinal motion (u_{rel}) in the north-south or meridional direction (**j**) as $-2\Omega u_{rel}\sin\varphi$ and in the vertical direction (**k**) as $2\Omega u_{rel}\cos\varphi$. The former is the well-known Coriolis effect for horizontal motion, the latter the so called "Eötvös effect" which explains why horizontal motions in the west or east directions make an object lighter or heavier. Meridional motion (v_{rel}) will only be deflected in the east-west direction (**i**) with $2\Omega v_{rel}\sin\varphi$ and so will also vertical motion (w_{rel}) with $2\Omega w_{rel}\cos\varphi$.

$$o = \alpha \frac{dds}{dt^2} + 2\alpha n \frac{dv}{dt} . \text{Sin.}^2 \theta + \alpha K \frac{ds}{dt} - g;$$

$$o = \alpha \frac{ddu}{dt^2} - 2\alpha n \frac{dv}{dt} . \text{Sin.} \theta . \text{Cos.} \theta + \alpha K \frac{du}{dt} - g \left(\frac{dy}{d\theta}\right)$$

$$o = \alpha \frac{ddv}{dt^2} . \text{Sin.} \theta + 2\alpha n \frac{du}{dt} . \text{Cos.} \theta - 2\alpha n \frac{ds}{dt} . \text{Sin.} \theta + \alpha K \frac{dv}{dt} . \text{Sin.} \theta - \frac{g}{\text{Sin.} \theta} . \left(\frac{dy}{d\omega}\right)$$

***Figure 8:* Laplace's derivation of the deflection in component form (p. 112)** *The entities* s, u *and* v *correspond to distances, their first and second derivative to velocities and accelerations. The rotational velocity, corresponding to Ω in modern notation is here represented by* n. *Laplace has also considered the possible influence due to friction which is seen in the* K*-terms. In the end the air friction has very little impact on the deflection.*

8. The interpretation of this $-2\ \Omega \times V_{REL}$ term

The Coriolis Effect was discovered in the 19th century mathematicians and physicists have since then struggled to find a good intuitive explanation: why a "– 2", and why a vectorial cross-product? So far nobody has really succeeded; so we must for the time being accept the term it at face value. On the other hand it is quite easy to understand what the **$-2\ \Omega \times V_{rel}$** term means in physical terms. Thanks to the vector notation that was developed just for the purpose of facilitating physical, intuitive understanding this can be stated as:

All relative motions V_{rel}, or components of relative motions, perpendicular to the rotational axis Ω will be deflected perpendicular both to the motion V_{rel} and the axis Ω to the right (for a counter clockwise rotation)[3]. And all motions, or components of motions, parallel to the rotational axis, will not be affected. (expl. in fig.9).

3. This is the case for Earth rotation – from west to east seen from above is counter clockwise (see also fig. 9 for that).

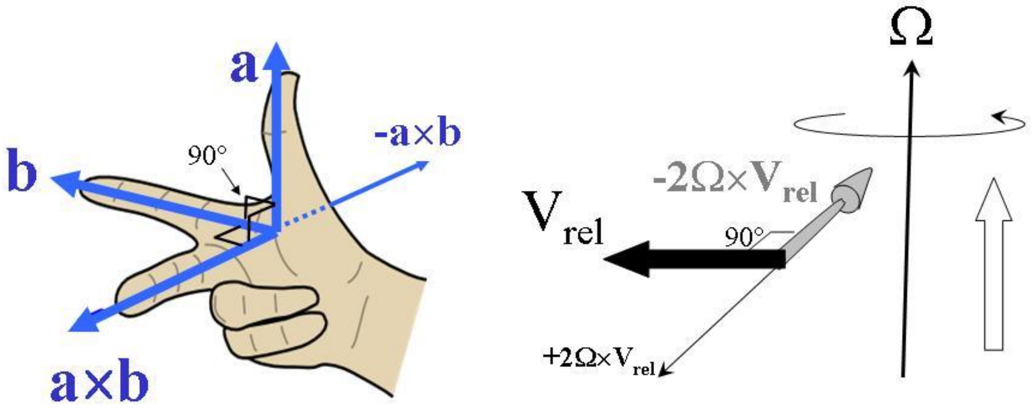

Figure 9: ***The relation between vectors* a, b *and their cross products* a×b *is facilitated through the "règle de la main droite" (left).*** *From this it is easy to see (right) how the relative motion* V_{rel} *, a vector perpendicular to the rotational axis Ω is deflected to* $-2\Omega\times V_{rel}$ *. Motions or components of motions, parallel to the rotational axis will not be affected, as indicated by the white vector to the far left.*

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More specifically, for vertical motion *w*, where *w* > 0 for upward motion (the time derivative of the distance to the Earth's centre) the velocity for free fall (w<0) can be written w= – g.t This can be split up into one component w·sinφ·g·t parallel to the Earth's axis, and another component, w·cosφ·g·t, perpendicular to it. Only the second component will be deflected to the right by the Coriolis effect.

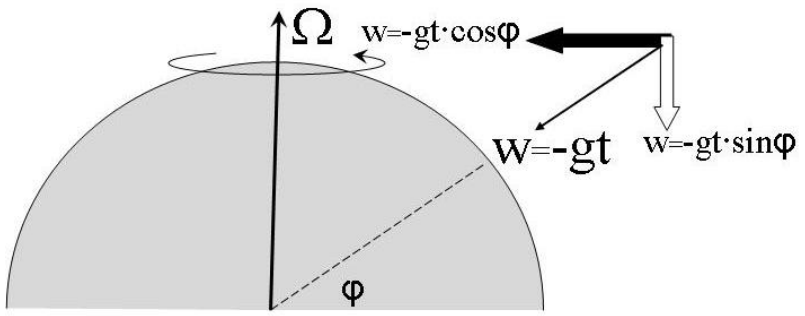

Fig. 10: ***The velocity of a falling body –*wg *can be split up into one component perpendicular to the Earth's axis, another parallel to it.*** *The first will be fully deflected, the second not.*

9. DEFLECTION ACCORDING TO THE CORIOLIS EFFECT

We can now, like Laplace (and Gauß) in 1803, calculate the deflection of the falling object. We first note that the time (t) it takes for an object to fall from an elevation (h) is

$$t = \sqrt{\frac{2h}{g}} \qquad (8a)$$

and it will then increase and when hitting the ground have a velocity

$$V_0 = g \cdot t \qquad (8b)$$

which can also be expressed as

$$V_0 = \sqrt{2hg} \qquad (8c)$$

or

$$h = \frac{V_0}{2g} \qquad (8d)$$

To gain clarity the derivations will be conducted for falls at the equator φ=0, where sinφ=0 and cosφ=1. For any calculation away from the equator, Ω can easily be replaced by Ωcosφ

For this vertically moving object we have, according to the cross product display in section 8 a deflection –2Ω×w, with w = –g·t; this can also be written as a second derivative of the position S

$$\frac{dS^2}{dt^2} = +2\Omega \cdot gt \qquad (9a)$$

Integrating *(9a)* over the time of the fall from a height *h*, with the initial horizontal velocity V_0, yields to

$$\frac{dS}{dt} = V_0 + \Omega \cdot gt^2 \qquad (9b)$$

Since the initial velocity V_0 = 0, further integration yields

$$S = S_0 + \Omega \cdot \frac{gt^3}{3} \qquad (9c)$$

Since our reference point is the base of the tower S_0 = 0 and *(8a)* yields

$$S = \frac{+\Omega}{3} \sqrt{\frac{8h^3}{g}} \qquad (10)$$

This corresponds exactly to Laplace's result p.115 (given that we took sinθ =1):

$$\frac{2\,nh}{3}.\ \text{Sin.}\ \theta.\sqrt{\frac{2h}{g}}.$$

This is again a rather mathematical explanation of the deflection with not much physical "feel". But, as often is the case in physics, there is more than one mathematical derivation for the same process or mechanism. We will make derivations which Isaac Newton and Johannes Kepler would have been able to do if they had come to think of it!

While the derivation of the deflection using the Coriolis effect was conducted in a relative frame of reference where we were positioned on the Earth and followed its rotation, we will now move into an absolute frame of reference, outside the Earth and look at the motion of the falling objects while they are transported around by the earth rotation.

10 . Deflection according to Newton's laws

For our first "Newtonian" derivation we make use of the insight, acquired already by the generation before Newton, that due to the curvature of the Earth the object will be affected by a component of gravity **g** directed "backward", which will delay the object's motion compared to the simplistic model in figure 1.

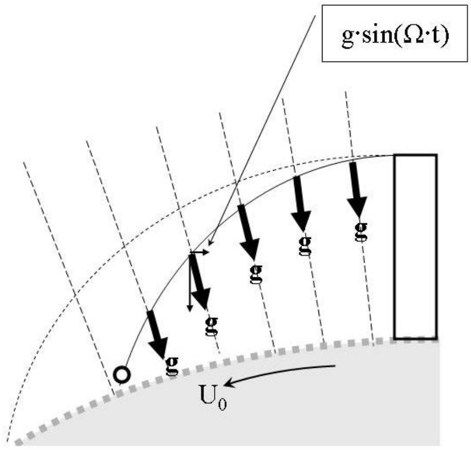

***Fig. 11:* In contrast to figure 1 where the gravitational force lines were parallel and the trajectory a parable, for radially converging force lines the trajectory becomes in principle an ellipse.**

This backward horizontal acceleration can be simplified because of the smallness of the angle Ω·t

$$a = -g \sin \Omega t \approx -g\Omega t = \frac{d^2 S}{dt^2} \qquad (11a)$$

which integrated yields

$$\frac{dS}{dt} = U - \frac{g\Omega t^2}{2} = \Omega(R+h) - \frac{g\Omega t^2}{2} \qquad (11b)$$

where U = Ω (R+h) is the Earth rotation at the elevation *h*. Finally we have

$$S = \Omega(R+h)t - \frac{g\Omega t^3}{6} \qquad (11c)$$

But a point at the ground, towards which the body is falling, is moving with a speed $U_0 = \Omega \cdot R < U$. This velocity difference yields $U - U_0 = \Omega h$ and when the body strikes the ground it undershoots by ΔS:

$$\Delta S = \Omega \cdot h \cdot t - \frac{g\Omega t^3}{6} \qquad (11d)$$

Inserting *(8a)* into (11d) yields

$$\Delta S = \Omega h \sqrt{\frac{2h}{g}} - \frac{\Omega}{6}\sqrt{\frac{8h^3}{g}} = \frac{\Omega}{3}\sqrt{\frac{8h^3}{g}} \qquad (12)$$

where the first term is the distance covered by the tower and the second term the slight deflection back from the tower.

If only Newton had more stubbornly considered the problem of the *deflection* of his famous falling apples, he would have been able to derive equation *(12)* already in the late 17th century and maybe also discovered the Coriolis effect. But he might not have been the first. Indeed it could have been done already some fifty years before by Johannes Kepler.

11 . Deflection according to Kepler's 2nd (area) law

Kepler's 2nd Law, that radius vector in equal times covers equal areas, was for long thought only to apply to celestial bodies, such as planets or comets. For most of the 1600s this law was not widely understood or accepted, not even by Isaac Newton. It was only during his work on "*Principia*" in the mid-1680s that he came to realise its validity, but also its shortcomings. It was, for example, only when he questioned one of the fundamental parts of Kepler's theory (that the trajectory of an orbiting planet has its focus, not in the centre of the sun as stated by Kepler, but in the *common centre of mass* of the sun and the orbiting planet), that he was able to formulate his third law.

Kepler had never extended his planetary laws to the neighbourhood of the Earth. It was not until the late 18th century that it was realised that his Second Law, the "Law of areas" could also be applied to also earthly objects and as such it became known as "*Conservation of Angular Momentum*".

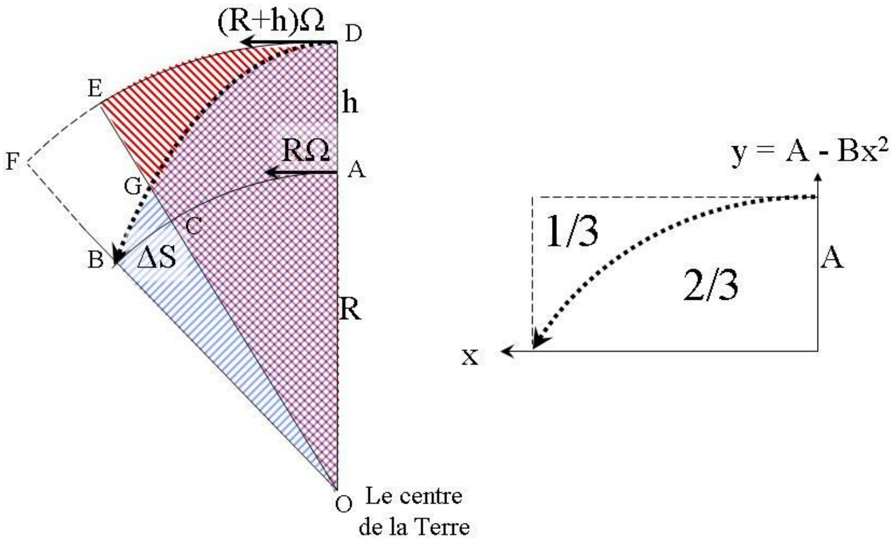

***Figure 12a:** Seen from outside the falling object follows the trajectory DGB while the top of the tower follows DE, and the base AC. **12b:** Any parable inscribed in a rectangle will cover 2/3 of its area.*

Our dropped object follows an absolute trajectory DGB while the base of the tower follows the trajectory AC (fig. 12a). According to Kepler's 2nd law the two areas ODBO (hatched blue and violet) and ODEO (hatched red and violet) are equal, and it is easily seen that the falling object moves ahead of the tower by the distance CB=ΔS.

Since the two motions share the area ODGO (hatched violet), the two not-shared, "leftover" areas DEGD (hatched only red) and area OGBO (hatched only blue) are equal in size. We then add to them the (white) area GEFB making area DFBD = area OEFO.

Considering h << R and the small angles involved (much smaller than in the figure) we approximate BG ≈ BC ≈ FE ≈ ΔS, essentially disregarding the area GBC ≈ 0 and treating GEFB as a rectangle with area ≈ h·ΔS and area OEFO ≈ ΔS (h+R)/2.

From our previous discussion (fig. 4), we know that the trajectory is an ellipse, but for the very short duration we are dealing with now, just a handful of seconds, we can treat the trajectory as a parable.

For the same reason we can treat DFBA as an rectangle with an area =(R+h)Ω·t. From the general rule that for a parable inscribed in a rectangle 2/3 of the rectangle's area is inside the parable, 1/3 outside (fig. 12b), we can approximately calculate the area DFBD = h·(R+h)·Ω·t/3, and we have

$$\frac{(R+h)\Delta S}{2} \approx \frac{h\cdot(R+h)\cdot\Omega\cdot t}{3} \qquad (13a)$$

which yields again with *(8)*

$$\Delta S \approx \frac{2h\cdot\Omega\cdot t}{3} = \frac{\Omega}{3}\sqrt{\frac{8h^3}{g}} \qquad (13b)$$

12. Laplace extension for deflection of vertically projected objects

When Laplace edited his Collected Works in 1805, he slightly modified his 1803 article, also including an analysis of the deflection of an object (like a cannonball) thrown vertically straight upward. This version can be found under the reference P.S.Laplace, *De la chute des corps qui tombent d'une grande hauteur*, Traité de Mécanique Céleste, T IV, Seconde Partie, Livre X, p. 294-305.[4]

294 MÉCANIQUE CÉLESTE,

CHAPITRE V.

De la chute des corps qui tombent d'une grande hauteur.

15. Un corps qui, partant de l'état de repos, tombe d'une grande hauteur, s'éloigne sensiblement de la verticale, en vertu du mouvement de rotation de la terre; cet écart bien observé est donc propre à manifester ce mouvement. Quoique la rotation de la terre soit maintenant établie avec toute la certitude que les sciences physiques comportent; cependant, une preuve directe de ce phénomène doit intéresser les géomètres et les astronomes. Afin que

Such experiments were indeed conducted in the early 1600's. In 1627 a German mathematician Joseph Furtenbach from Ulm fired cannonballs vertically and, being sure that they would not come straight down, immediately after the shots climbed up and sat on the cannon's muzzle. The same experiment was

4. The 1805 version (Google Books) corresponds in his first part (§15, p. 294-302) to the 1803 article (with some revisions); the second part (§16, p. 302-305) is an addendum where Laplace discusses the problem of the deflection.

carried out 1634 also by Mersenne, at the request of Descartes. Galileo's students in Florence conducted in the 1650's an experiment where the cannon was mounted on a wagon drawn by six horses in a rapid speed to see if the motion of the wagon would make the balls drop differently compared to when the wagon was at rest.

The results were generally inconclusive. Cannonballs were not uncommonly lost, and this gave rise to the belief that they broke away from the earth's gravity and travelled into space never to return. It is more likely that they were carried away by the strong winds often found high up in the atmosphere.

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We now repeat this derivation in the "Coriolian" way Laplace did in 1803 and then how it could have been done in a "Newtonian" and "Keplerian" fashion. First the "Coriolian" with an initial upward velocity V_0 and where the Coriolis deflection at time t is:

$$\frac{dS^2}{dt^2} = -2\Omega \cdot (V_0 - gt) \qquad (14a)$$

when integrated becomes

$$\frac{dS}{dt} = -2\Omega \cdot V_0 \cdot t + \Omega g t^2 \qquad (14b)$$

and finally

$$S = -\Omega \cdot V_0 \cdot t^2 + \frac{\Omega g t^3}{3} \qquad (14c)$$

The expression of time as a function of V_0 as presented in *(8b)* must be doubled since the time it takes for the projectile to reach height *h* and then descend back to earth is double compared to a fall from height *h*. This makes

$$t = \frac{2V_0}{g} \qquad (15)$$

and we get a deflection

$$S = \frac{-\Omega \cdot V_0 \cdot 4V_0^{\,2}}{g^2} + \frac{\Omega g 8V_0^{\,3}}{3g^3} = -\frac{4}{3}\frac{\Omega V_0^{\,3}}{g^2} \qquad (16)$$

Since S < 0, the deflection is to the west. This result needs a further explanation. For a projectile fired upward in the air and then falling back, one would expect that the deflection during the falling back part of the motion should cancel the deflection during the upward part.

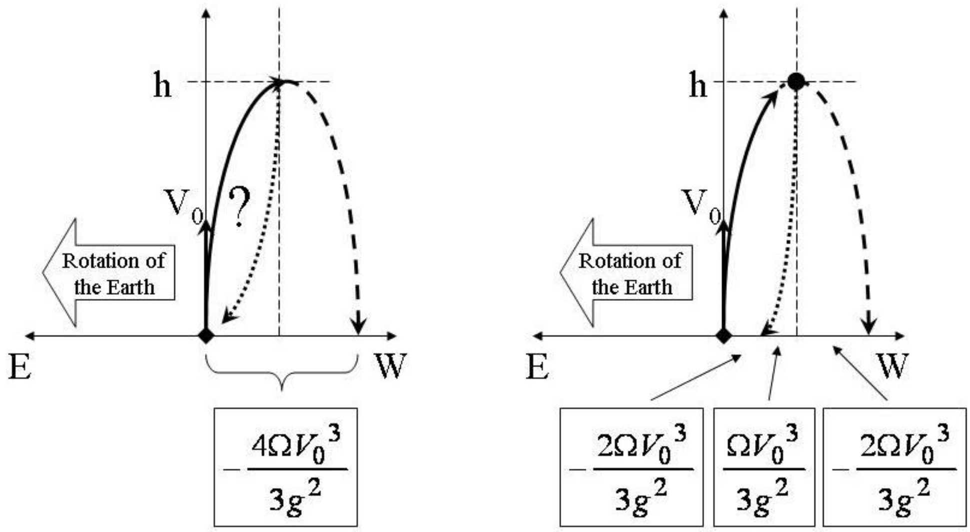

Figure 13: *The relative trajectory, relative to the cannon and the ground, of a vertically projected object with initial velocity V_0.* ***Left:*** *Why is the object not deflected back when it falls back to the Earth, as would an object released from rest at the same height?* ***Right:*** *An object released from the same height would fall eastward, but the eastward deflection is not large enough to compensate for the initial westerly deflection. Further, since the projected body is moving westward at its highest point and the released body is at rest, this further adds to their separation.*

Let us imagine that our upward projected object, when reaching its highest point, by some coincidence, should be close to another projectile, released from rest into a free fall. Since both are on their way down, we might expect them to be close and follow each other. But the projected object is at this highest point, due to the Coriolis effect, not at rest but in horizontal motion to the west. Further, the eastward deflection of the objects starting from rest will anyhow only be half of the westward deflection it is supposed to compensate for (fig. 13).

13. The deflection of vertically projected objects according to Newton's laws

And now, in an absolute frame of reference, according to Newton's laws, the backward deflection due to the curvature of the earth's surface as previously discussed:

$$\frac{d^2S}{dt^2} = -g\Omega t \qquad (17a)$$

$$\frac{dS}{dt} = \Omega R - \frac{g\Omega t^2}{2} \qquad (17b)$$

$$S = \Omega Rt - \frac{g\Omega t^3}{6} \qquad (17c)$$

With the expression for the time *(15)* as above

$$S = \Omega Rt - \frac{4}{3}\frac{\Omega V_0{}^3}{g^2} \qquad (18)$$

So while the cannon travels a distance ΩRt carried by the Earth rotation, the ball falls slightly behind ($S < \Omega Rt$).

14. The deflection of vertically projected objects according to Kepler's 2nd law

Using Kepler's 2nd Law of equal areas we can immediately see that the position where the object is projected vertically upward will overtake the object itself since the area OADBO = area OACO. Both have the area OABO (violet) in common so we can concentrate on (red) area OBCD=R·ΔS/2 and (blue) area ADBA=2Ω·R·t·h/3 which, although an ellipse, can be approximated as a parable inscribed in a rectangle with base=2Ω·R·t and height=h.

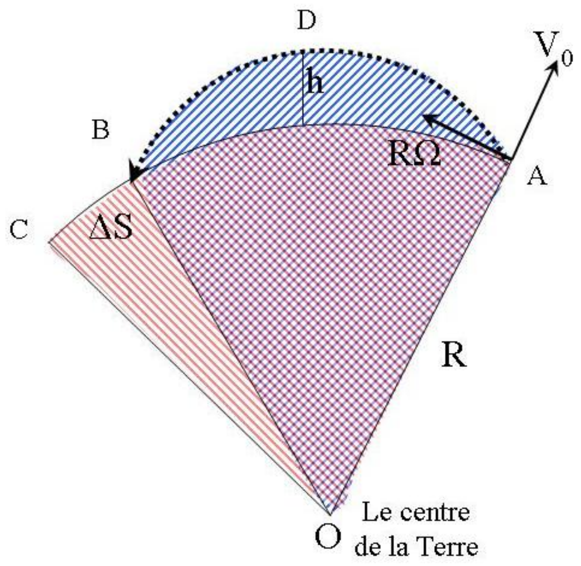

***Figure 14:* The trajectory of the vertically projected body ADB seen from an absolute frame of reference.** *During the same time the position of the launch (the "cannon") has travelled the longer distance ABC. In both cases the "radius vector" covers equal areas OADB=OABC. The trajectory ADB is in reality an ellipse but can here be approximated by a parable.*

Using *(8a)* and *(15)* we integrate over two time periods we find once again the same formula:

$$S = \Omega R t - \frac{4}{3}\frac{\Omega V_0^{\ 3}}{g^2}$$

In those days a cannon had a typical exit velocity (V_0) of 400 m/s. With an angle of 45° this would mean both a horizontal velocity and a vertical initial velocity $V_{0H} = V_{0V} = 283$ m/s. It will take the grenade 29 seconds to reach its highest point just above 4 km and almost 58 seconds to reach its target (assuming no air friction) which will be little more than 16 km away. The grenade will undershoot the target by 45 m. So far in the equator region, where $2\Omega\sin\varphi$ is null, and we therefore do not have any horizontal, sideway, deflection.

If we now move to a north Italian latitude of 43° where $\Omega\cos\varphi = 0.53\cdot10^{-4}$ /sec the projectile will undershoot the target by 17 m. But with a value for $2\Omega\sin\varphi = 1.00\cdot10^{-4}$ /sec there will be a sideways deflection of 47 m.

15. Other aspects of Laplace's derivations

As mentioned in the introduction, Laplace made his derivation very consciously in order to also establish the effects of friction and any possible southerly deflection. We will not go into these details, only mention that friction according to Laplace (and subsequent derivations) does not seem to have any significant impact on the deflection.

Concerning a possible southerly, or rather equatorward deflection, was, as we can read in Laplace's article, a matter of controversy already 200 years ago, Laplace's calculations showed no such deflection but Gauß' calculations did; this contributed to keep the issue controversial into our times. Part of the problem is to define "southerly"; is it in relation to the geographical latitude or the geocentric? Laplace and Gauss were not aware of later geodetic work that defined the shape of the earth more exactly than was the case in their times. Today's agreed wisdom is that the deflection does not exist or is so small that it is over-shadowed by necessary mathematical approximations. Finally, experiments in modern times have so far not found any detectable southerly deflection.

16. Why did it take almost 200 years?

We have thus shown that correct expressions for the deflection of a falling object could have been derived already 100 years earlier by Newton or almost 200 years earlier by Kepler. So why didn't they do it?

In Newton's case it was because he was never asked. Deflection of falling bodies had been, as we have seen, high on the scientific agenda in the 17th century, but simplistic calculations (fig. 1) had yielded very small values. Hooke's experiment had shown a very great spread of measurements and the matter had been regarded as impossible to pursue scientifically.

But also Benzenberg's measurements (fig. 5) showed large spread and so did Ferdinand Reich's measurements in a later experiment 1834. By that time the understanding of error statistics had developed, and it was seen a natural thing to compute averages of measurements. Well into the late 18th century scientists, in particular astronomers, had the habit of trying to find out which of several observations or measurements they had done, was "the best" one. Combining observations would, so it was thought, add the errors.

Thanks to Laplace, Gauß, Legendre[5] and others, it become established that due to the random nature of errors, combining observation would to a large degree cancel out the errors yielding averages that would be more accurate than any randomly chosen observation.

The reason why Kepler would not have been able to apply his 2nd law on the problem was because it was considered to apply only to celestial motions, not terrestrial. That became clear only after the publication of Newton's "Principia". Before that there were even doubts if the law was correct, since it had been derived from observations from only one planet, Mars, with an unusually eccentric orbit.

Finally, Laplace and Gauß managed to calculate the deflection after having derived what we now regard as the "Coriolis Effect", some 30 years before Coriolis. Why did they, or one of them, not get the credit of later generations? One reason might be that Coriolis's work dealt with the dynamics of machines and appeared as less far-fetched than Laplace's and Gauß's. But more likely, the interest in the behaviour of relative motion in rotating system had its "breakthrough" after Foucault's famous experiment in 1851. In the ensuing discussions Coriolis's 1835 paper happened to be more in the scientists mind because the attention it had contracted a few years earlier. French mathematician Joseph Bertrand had 1847-48 claimed that Coriolis had plagiarized results already derived by Alexis Clairault some 100 years earlier. There might be reasons to come back to this story in a later contribution.

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(August 2014)

5. About Legendre and his method of least squares for the approximate solution of overdetermined systems, see *BibNum* analysis by J.-J. Samueli, August 2010 (online).